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［論 文］

A Note on Static and Dynamic Characteristics of Macro Models*

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R. J. Ball and R. G. Bodkin developed price multipliers along with income multipliers, in order to examine the effects of policies on real income and the price level based on essentially Keynesian theory by the techniques of comparative statics.[1] The purpose of this paper is to extend and modify their research first to open systems with different supply functions of labor from their model and second examining whether those systems are stable or not dynamically. We deal with the problems faced by the policy maker concerned with the impacts of his actions and of the balance of payment on the general price level. Dynamic aspects of the inflationary processes are given by the phase diagrams of two markets and stability conditions in the cases with more than two markets without knowing the absolute values of each partial differential, because we can only gain sign of each value.

Section I, III, and V deal with cases of over full employment and section II, IV, and VI under full employment.

I

Our model is described as follows. Here we are not going to include foreign trade. We have as accounting identity,

$$Y=C+I+G \tag{1}$$

where Y is national income, C is consumption expenditures, I is net investment and G is government expenditures, all of which is considered in real terms.

Consumption is considered to depend upon the level of real income Y, the real initial stock of money M_0/P, the rate of interest i, which gives

$$C=C(Y,i,M_0/P)^{1)} \tag{2}$$

Net investment is also treated to depend on these same variables.

$$I=I(Y,i,M_0/P) \tag{3}$$

Government expenditures is assumed to be $g\%$ of national income Y.

$$G=gY \tag{4}$$

This is a short period analysis, therefore the effect of the stock of productive equipment is taken as given and technological progress is negligible.

We have the production function.

$$Y=f(N), \qquad f'(N)>0, \qquad f''(N)<0 \tag{5}$$

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1) We use initial stock money M_0 here, this has different notation with money supply M which will be described later at [8].
If M is taken instead of M_0, we can't decide the sign of di/dM.

The demand of labor is decided when the marginal productivity of labor $f'(N)$ is equal to the rate of real wage with the assumption of profit maximization.

$$f'(N)=W/P \tag{6}$$

In the classical system the supply of labor is supposed to behave rationally. In the same way, that the supply of any commodity depends on the labor is taken to depend not on the money wage, but on the real wage.

This function of supply of labor is the same even in the Keynesian system on condition over full employment level according to F. Modigliani. He expressed the Keynesian assumptions concerning the supply of labor in functional form.[2]

$$W=aW_0+bF(N)P$$

where a and b are functions of N, W, P characterised by the following properties:

$$a=1,\quad b=0,\quad \text{for } N\leqq N_0$$
$$a=0,\quad b=1,\quad \text{for } N>N_0$$

where N is said to be full employment and W_0 is the historically ruling wage rate. Therefore, under full employment (including $N=N_0$), the Keynesian system is different from the classical one, but over full employment Keynesian system is the same with the classical one.

The supply of labor can be regarded as the function of real wage rate.

$$U=F(W/P)$$

We define the reverse function of F as S.

$$W/P=F^{-1}(N)=S(N) \tag{7}$$

The demand of money is taken to depend upon the level of money income and the rate of interest. Let the money supply be M, then the equilibrium of money market is described by

$$M=L(PY,i) \tag{8}$$

Then, considering the model as a whole, we have eight equations in eight endogeneous variables that is, Y, C, I, G, i, N, W, P, and the number of equations is equal to that of variables. Exogeneous variables in our model are M, g.

Now we are going to simplify these eight equations with eight variables into three equations with three unknowns.

$$\begin{cases} C(f(N),i,M_0/P)+I(f(N),i,M_0/P)+gf(N)-f(N)=0 & (9)\\ L(Pf(N),i)-M=0 & (10)\\ f'(N)-S(N)=0 & (11)\end{cases}$$

Total differentiation of this system with respect to M yields,

$$\begin{cases} -(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2\,dP+(Ci+Ii)di-f'(N)(1-g-C_{f(N)}-I_{f(N)})dN=0^{2)} & (12)\\ L_{Pf(N)}f(N)\,dP+Lidi+Pf'(N)L_{Pf(N)}dN=dM & (13)\\ (f''(N)-S'(N))dN=0 & (14)\end{cases}$$

According to the Cramer's rule, we obtain,

$$\frac{dP}{dM}=\frac{1}{D}\begin{vmatrix} 0 & Ci+Ii & -f'(N)(1-g-C_{f(N)}-I_{f(N)})\\ 1 & Li & Pf'(N)L_{Pf(N)}\\ 0 & 0 & f''(N)-S'(N)\end{vmatrix}$$
$$=\frac{1}{D}(f''(N)-S'(N))(-Ci-Ii)>0$$

where

2) $C_{\frac{M}{P}}$ means the partial differentiation of C with respect to M/P. We use this notation from now on.

$$D=\begin{vmatrix} -(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2 & Ci+Ii & -f'(N)(1-g-C_{f(N)}-I_{f(N)}) \\ L_{Pf(N)}f(N) & Li & Pf'(N)L_{Pf(N)} \\ 0 & 0 & f''(N)-S'(N) \end{vmatrix}$$
$$=(f''(N)-S'(N))(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2 Li-(Ci+Ii)L_{Pf(N)}f(N))<0$$

Here we can assume following conditions from the standard theory.

$$f''(N)<0,\quad S'(N)>0,\quad Ci<0,\quad Ii<0,\quad Li<0,\quad C_{\frac{M}{P}}>0,\quad I_{\frac{M}{P}}>0,\quad L_{Pf(N)}>0.$$

then $D<O$

$$\frac{di}{dM}=\frac{1}{D}(f''(N)-S'(N))(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2)<0$$

Money supply has negative effect on the rate of interest.

$$\frac{dN}{dM}=0$$

By the same way,

$$\frac{dP}{dg}=\frac{1}{D}(f''(N)-S'(N))(-f(N)Li)i>0$$
$$\frac{di}{dg}=\frac{1}{D}(f''(N)-S'(N))(+L_{Pf(N)}f^2(N))>0$$
$$\frac{dN}{dg}=0$$

The expansion of government expenditures raises price and the rate of interest.

II

Mentioning just before, under full employment level of Keynesian theory by the F. Modigliani's formulation, the supply function of labor is regared as,

$$W=W_0$$

therefore, instead of equation (11) we can get,

$$Pf'(N)-W_0=0 \tag{11)'}$$

and equation (14) shall be

$$f'(N)dp+Pf''(N)dN=0 \tag{14)'}$$

$$\frac{dP}{dM}=\frac{1}{D'}\begin{vmatrix} 0 & Ci+Ii & -f'(N)(1-g-C_{f(N)}-I_{f(N)}) \\ 1 & Li & Pf'(N)L_{Pf(N)} \\ 0 & 0 & Pf''(N) \end{vmatrix}$$
$$=\frac{1}{D'}Pf''(N)(-Ci-Ii)$$

where

$$D'=\begin{vmatrix} -(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2 & Ci+Ii & -f'(N)(1-g-C_{f(N)}-I_{f(N)}) \\ L_{Pf(N)}f(N) & Li & Pf'(N)L_{Pf(N)} \\ f'(N) & 0 & Pf''(N) \end{vmatrix}$$
$$=-(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2 LiPf''(N)+(Ci+Ii)Pf'(N)L_{Pf(N)}f'(N)$$
$$-f'(N)(1-g-C_{f(N)}-I_{f(N)})Lif'(N)-Pf''(N)L_{Pf(N)}f(N)(Ci+Ii)<0$$

$$\frac{dP}{dM}=\frac{1}{D'}Pf''(N)(-Ci-Ii)>0$$
$$\frac{di}{dM}=\frac{1}{D'}(-C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2\,Pf''(N)+f'(N)(1-g-C_{f(N)}-I_{f(N)})f'(N)<0$$
$$\frac{dN}{dM}=\frac{1}{D'}f'(N)(Ci+Ii)>0$$

$$\frac{dP}{dg}=\frac{1}{D'}(-f(N)LiPf''(N))>0$$
$$\frac{di}{dg}=\frac{1}{D'}(f(N)L_{Pf(N)}f(N)Pf''(N)-f'(N)Pf'(N)L_{Pf(N)})>0$$
$$\frac{dN}{dg}=\frac{1}{D'}f(N)Lif'(N)>0$$

III

Next we are going to deal with the dynamic characteristics of the above formulated systems, by the technique of dynamization.

In the case of demand of good exceeding the supply of good, price of this good is assumped to rise and vice verce.

$$\dot{P}=X(C(f(N),i,M_0/P)+I(f(N),i,M_0/P)+gf(N)-f(N)) \quad (15)$$
$$X'>0, \quad X(0)=0$$

If the demand of money exceed the supply of money we assume that the rate of interest tends to rise and vice verce.

$$\dot{i}=Y(L(Pf(N),i)-M) \quad (16)$$
$$Y'>0, \quad Y(0)=0$$

When the rate of real wage by the demand function of labor exceeds the rate of real wage by the supply function of labor, we assume that labor increases and when opposite condition occures, labor decreases.

$$\dot{N}=Z(f'(N)-S(N))^{3)} \quad (17)$$
$$Z'>0, \quad Z(0)=0$$

Assuming that money market does not have any kind of disequilibrium and neglecting the effect of money market on total system, that is, (15), (16), (17), general equilibrium of system consisting of two markets, goods and labor markets, can be considered and phase diagram of these markets can be drawn.

Differentiating partially the equation (15),

$$\frac{\partial\dot{P}}{\partial P}=X'\cdot-(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2<0$$

can be obtained.

On condition that $\dot{P}=0$

$$-(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2\,dP-f'(N)(1-g-C_{f(N)}-I_{f(N)})dN=0$$
$$\frac{dN}{dP}=-(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2/f'(N)(1-g-C_{f(N)}-I_{f(N)})<0$$

From equation (17),

$$\frac{\partial\dot{N}}{\partial N}=Z'(f(N)-S(N))<0$$

On condition that $\dot{N}=0$

$$(f''(N)-S'(N))dN=0$$

We can get the Figure 1 according to the conditions described above.

Neglecting the effect of labor market, from equation (15)

$$\frac{di}{dP}=-(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2<0$$

3) The equilibrium of (15), (16) is given by the adjustment of price, but in the case of (17), by quantity of labor. We assume that both adjustment variables lead the system of equations to the same equilibrium.

and from equation (16),

$$\frac{\partial \dot{i}}{\partial i}=Y'Li<0.$$

On condition that $\dot{i}=0$,

$$\frac{di}{dP}=-L_{Pf(N)}/Li>0$$

These conditions gives the phase diagram Figure 2.[4)]

Figure-1

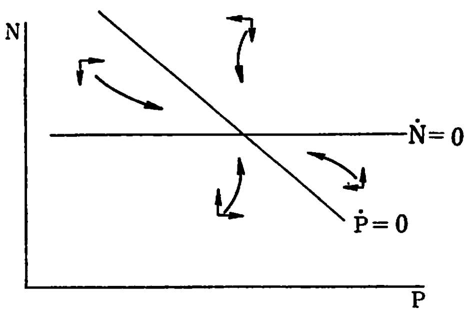
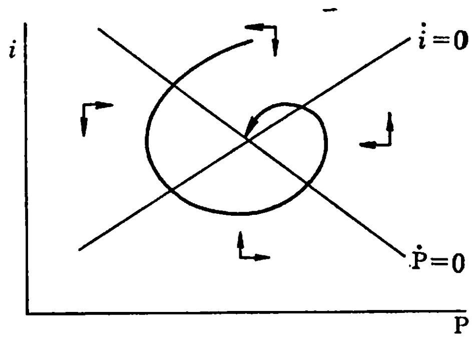


Figure-2



We can not examine the joint stability of three markets by the method which we used till now. Then following technique can be applied to the total system. The equations (15), (16), (17) can be developed by using Taylor's series near the equilibrium points, $\bar{P}$, $\bar{i}$, $\bar{N}$, of which P, i, N are, respectively.[3]

$$\begin{cases} \dot{P}=X'(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2(P-\bar{P})-(Ci+Ii)(i-\bar{i})-f'(N)(1-g-C_{f(N)}-I_{f(N)})(N-\bar{N})) \\ \dot{i}=Y'(L_{Pf(N)}f(N)(P-\bar{P})+Li(i-\bar{i})+LPf(N)(N-\bar{N})) \\ \dot{N}=Z'((f''(N)-S'(N))(N-\bar{N})) \end{cases}$$

We change variables $P-\bar{P}, i-\bar{i}, N-\bar{N}$, into P', i', N', respectively. The stability of this system can be examined by forming the following characteristic equation.

$$\begin{cases} \dot{P}'=X'(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2\,P'+(Ci+Ii)i'-f'(N)(1-g-C_{f(N)}-I_{f(N)})N') & (18) \\ \dot{i}'=Y'(L_{Pf(N)}f(N)P'+Lii'+L_{Pf(N)}Pf(N)N') & (19) \\ \dot{N}'=Z'((f''(N)-S'(N))N') & (20) \end{cases}$$

The characteristic equation is

$$\begin{vmatrix} -(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2-Q & Ci+Ii & -f'(N)(1-g-C_{f(N)}-I_{f(N)}) \\ L_{Pf(N)}f(N) & Li-Q & L_{Pf(N)}Pf(N) \\ 0 & 0 & f''(N)-S'(N)-Q \end{vmatrix}=0$$

By some manipulations

$$\begin{aligned} &Q^3-Q^2(f'(N)-S''(N)-(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2+Li) \\ &\quad -Q((C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2\,Li+(f''(N)-S'(N))((C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2-Li) \\ &\quad +(Ci+Ii)L_{Pf(N)}f(N)+(C_{\frac{M}{P}}+I_{\frac{M}{P}})M_0/P^2\,Li(f''(N)-S'(N)) \\ &\quad +(Ci+Ii)(f''(N)-S'(N))L_{Pf(N)}f(N)=0 \end{aligned}$$

Here we remember the Routh theorem.[4]

Given the characteristic equation

$$a_0X^n+a_1X^{n-1}+a_2X^{n-2}+a_3X^{n-3}+\cdots+a_n=0$$

4) By making characteristic equation, from (15), (16) about dP, di, we get $Q^2+QA+B=0$ and $A>0$, $B>0$. Therefore this system is stable.

The Routh theorem states that the real part of all of the roots will be negative if and only if the following n determinants are all positive, which means the original system of differential equations is stable.

$$a_1 \quad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} \quad \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

This theorem can be applied to our characteristic equation.

$$a_1 = -\left(f''(N) - S'(N) - (C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M_0}{P^2} + Li\right) > 0$$

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = a_1a_2 - a_0a_3$$

$$= \left(f'' - S' - (C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M_0}{P^2} + Li\right)\left((C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M_0}{P^2}Li + (f''-S')(C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M_0}{P^2} - Li\right)$$

$$+ (Ci + Ii)L_{Pf(N)}f - (C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M}{P}Li(f''-S') - (Ci+Ii)(f''-S')L_{Pf(N)}f$$

$$= (f''-S')\left((C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M_0}{P^2} - Li\right) + \left(-(C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M_0}{P^2} + Li\right)\left((C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M}{P}Li + (f''-S')\right)$$

$$\times\left((C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M_0}{P^2} - Li + (Ci+Ii)L_{Pf(N)}f\right) > 0$$

$$\begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0 \qquad (\because \ a_3 > 0)$$

Therefore the nature of the time path of the variables of equations (18), (19), (20) are stable jointly.

IV

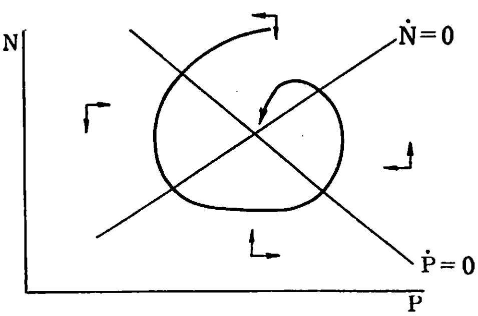
We examine the dynamic characteristics of the Keynesian system with the condition of the supply function of labor being under full employment by the equation (14)′ instead of (14). The equilibrium condition between price and labor is different from the system analyzed in the previous section.

from the equation (14)′

$$\frac{dN}{dP} = f'(N) / -Pf''(N) > 0$$

with the condition $\dot{N}=0$ and with the results obtained in section III, also, the following phase diagram is given.

Figure-3



The equilibrium of three markets is analyzed.

$$\begin{vmatrix} -(C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M_0}{P^2} - Q & Ci - Ii & -f'(N)(1 - g - C_{f(N)} - I_{f(N)}) \\ L_{Pf(N)}f(N) & Li - Q & L_{Pf(N)}Pf'(N) \\ f'(N) & 0 & Pf''(N) - Q \end{vmatrix} = 0$$

$$Q^3 - Q^2\left(Pf''(N) - (C_{\frac{M}{P}} + I_{\frac{M}{P}})\frac{M_0}{P^2} + Li\right)$$

$$-Q\Big((C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}Li+Pf''\Big((C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}-Li\Big)-f'^2(1-g-C_{f(N)}-I_{f(N)})$$
$$+(Ci+Ii)L_{Pf(N)}f\Big)+(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}LiPf''-(Ci+Ii)L_{Pf(N)}Pf'$$
$$-f'^2(1-g-C_{f(N)}-I_{f(N)})Li+(Ci+Ii)Pf''L_{Pf(N)}f=0$$
$$|a_1|=-\Big(Pf''-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+Li\Big)>0$$
$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}=a_1a_2-a_0a_3$$
$$=\Big(Pf''*(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P}+Li\Big)\Big((C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P}Li+Pf''\Big((C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P}-Li\Big)$$
$$-f'^2(1-g-C_{f(N)}-I_{f(N)})+(Ci+Ii)I_{Pf(N)}Pf'\Big)-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P}LiPf''$$
$$+(Ci+Ii)L_{Pf(N)}Pf'^2+f'^2(1-g-C_{f(N)}-I_{f(M)})Li-(Ci+Ii)Pf''L_{Pf(N)}f$$

We can not decide the sign of this equation without knowing the value of each differential. But, if we can adopt the result obtained in section II, which is $dP/dM>0$, that is, the quantity theory of prices is valid and interest elasticity of consumption and investment can be regarded as zero, $Ci+Ii\fallingdotseq 0$ and $a_1a_2-a_0a_3>0$. Then above system would be stable. Because only the term, $(Ci+Ii)L_{Pf(N)}Pf'$ remains negative.

V

In this section, foreign trade is added to our system and some terms modified. The balance of payment is treated as the difference between exports and imports, which does not consider the effect of capital movement among nations.

$$B=E\Big(\frac{PR}{p}\Big)-T\Big(\frac{PR}{p}\Big) \qquad (21)$$

where R is the rate of foreign exchange, E is export, T is import and p is world price.

With modifying equations in section I and II, new open system is given as follows.

$$\begin{cases} C\Big(f(N),\ i,\ \frac{M_0}{P}\Big)+I\Big(f(N),\ i,\ \frac{M_0}{P}\Big)+gf(N)-f(N)+E\Big(\frac{PR}{p}\Big)-T\Big(\frac{PR}{p}\Big)=0 & (22) \\ L(Pf(N),\ i)-M=0 & (23) \\ f'(N)-S(N)=0 & (24) \\ E\Big(\frac{PR}{p}\Big)-T\Big(\frac{PR}{p}\Big)-B=0 & (25) \end{cases}$$

We shall introduce an assumption that the exchange rate is flexible in this system and $B=0$.

Total differentiation of this system with respect to M is.

$$\begin{cases} \Big(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\Big)dP+(Ci+Ii)di-f'(1-g-C_{f(N)}-I_{f(N)})dN \\ \quad +\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})dR=0 \\ L_{Pf(N)}f(N)dP+Lidi+L_{Pf(N)}Pf'(N)dN=dM \\ (f''(N)-S'(N))dN=0 \\ \frac{R}{p}(E_{\frac{PR}{p}}-I_{\frac{PR}{p}})dP+\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})dR=0 \end{cases}$$

By the Cramer's rule,

$$\frac{dP}{dM}=\frac{1}{D_1}\begin{vmatrix} 0 & Ci+Ii & -f(N)(1-g-C_{f(N)}-I_{f(N)}) & \frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{RP}{p}}) \\ 1 & Li & L_{Pf(N)}Pf'(N) & 0 \\ 0 & 0 & f''(N)-S'(N) & 0 \\ 0 & 0 & 0 & \frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) \end{vmatrix}$$

$$=\frac{1}{D}\left((-Ci-Ii)\,(f''(N)-S'(N))\,(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\frac{P}{p}\right)$$

where

$$D_1=\begin{vmatrix} -(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) & Ci+Ii & -f(N)(1-g-C_{f(N)}-I_{f(N)}) & \frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) \\ L_{Pf(N)}(N) & Li & L_{Pf(N)}Pf(N) & 0 \\ 0 & 0 & f''(N)-S(N) & 0 \\ \frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) & 0 & 0 & \frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) \end{vmatrix}$$

$$=-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}Li(f''-S')\frac{P}{p}(E_{\frac{PR}{p}}-I_{\frac{PR}{p}})-L_{Pf(N)}f(Ci+Ii)(f''-S')\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})>0$$

$$\therefore\quad \frac{dP}{dM}>0 \qquad (\because\ E_{\frac{PR}{p}}<0,\qquad T_{\frac{PR}{p}}>0)$$

and

$$\frac{di}{dM}=\frac{1}{D_1}\left(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M}{P^2}(f''-S')(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\frac{P}{p}\right)<0$$

$$\frac{dN}{dM}=0$$

$$\frac{dR}{dM}=\frac{1}{D_1}\left(\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})Li(f''-S')\,(C_{\frac{PR}{p}}+I_{\frac{PR}{p}})\frac{1}{P}+(Ci+Ii)(f''-S')\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{RP}{p}}\right)>0$$

Here, we assume that $B\fallingdotseq 0$ and R is flexible. That is, we have the system of crawling peg. By the same way with respect to exogeneous variable B,

$$\frac{dP}{dB}=\frac{1}{D_1}\left(-\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})Li(f''-S')\right)>0$$

$$\frac{di}{dB}=\frac{1}{D_1}\left(\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})(f''-S')L_{Pf(N)}f(N)\right)>0$$

$$\frac{dN}{dB}=0$$

$$\frac{dR}{dB}=\frac{1}{D_1}\left(\left((-C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\right)Li(f''-S')-L_{Pf(N)}f(Ci+Ii)(f''-S')\right)<0$$

The increase of the balance of payment raises price of commodity and lowers the rate exchange.

Solving for the changes by dg, we obtain,

$$\frac{dP}{dg}=\frac{1}{D_1}\left(\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})-f(N)L(f''-S')\right)>0$$

$$\frac{di}{dg}=\frac{1}{D_1}\left(\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})f(N)(f''-S')L_{Pf(N)}f(N)\right)>0$$

$$\frac{dN}{dg}=0$$

$$\frac{dR}{dg}=f(N)Li(f''-S')\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})>0$$

The increase of government raises price and interest rate and lowers the rate of exchange.

And P multiplier,

$$\frac{dP}{dp}=\frac{di}{dp}=\frac{dN}{dp}=0$$

$$\frac{dR}{dp}=\frac{1}{D_1}\Big(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}Li(f''-S')\frac{P}{p^2}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})-L_{Pf(N)}f(N)(Ci+Ii)$$
$$\times(f''-S')\frac{P}{p^2}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\Big)>0$$

If the world price rises, the rate of exchange rises and vice verce.

We also examine the dynamic characteristics of this open system by the technique of dynamization like done in section III. Here, we add one more assumption that in the case of export exceeding import, the rate of exchange rises and opposite direction occurs with the opposite change.

$$\dot{P}''=X'\Big(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\Big)P''+(Ci+Ii)i''-f'(1-g-C_{f(N)}-I_{f(N)})N''$$
$$+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})R''\Big)$$
$$X'>0,\quad X'(0)=0 \qquad (26)$$
$$\dot{i}''=Y'(L_{Pf(N)}f(N)P''+Li\ i''+L_{Pf(N)}Pf(N)N'')$$
$$Y'>0,\quad Y'(0)=0 \qquad (27)$$
$$\dot{N}''=Z'(f''(N)-S'(N)N'')$$
$$Z'>0,\quad Z'(0)=0 \qquad (28)$$
$$\dot{R}''=W'\Big(\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})P''+\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})R''\Big)$$
$$W'>0,\quad W'(0)=0 \qquad (29)$$

If we can neglect the effect of money and labor markets, the equilibrium between price and the rate of exchange is given like follows. From equation (26),

$$\frac{\partial\dot{P}}{\partial P}=-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}<0,$$

Conditioning that $\dot{P}=0$

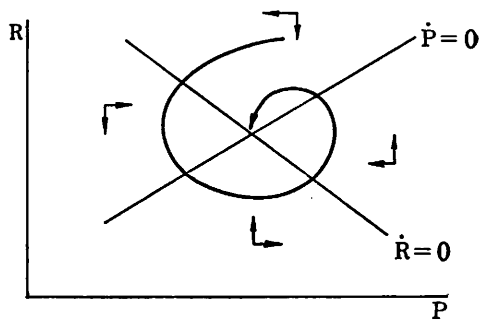
$$\frac{dR}{dP}=\frac{-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}}{\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})}>0,$$

From equation (29),

$$\frac{\partial\dot{R}}{\partial R}=\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})<0$$

Conditioning that $\partial\dot{R}=0$,

Figure-4



$$\frac{dP}{dR}=\frac{-\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})}{\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})}<0$$

Next, we deal with the problem of the equilibrium of four markets including foreign trade by the same method used in section III. The characteristic equation is given,

$$\begin{vmatrix} -(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})-Q & Ci+Ii & -f'(N)(1-g-C_{f(N)}-I_{f(N)}) & \frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) \\ L_{Pf(N)}f(N) & Li-Q & L_{Pf(N)}Pf(N) & 0 \\ 0 & 0 & f''(N)-S'(N)-Q & 0 \\ \frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) & 0 & 0 & \frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})-Q \end{vmatrix}$$

$$=0$$

For simplification of calculation, we change each term of the above determinant as follows.

$$\begin{vmatrix} A-Q & B & C & D \\ E & F-Q & G & 0 \\ 0 & 0 & H-Q & 0 \\ I & 0 & 0 & D-Q \end{vmatrix}=0$$

Here $A, B, C, D, F, H, I, <0,\quad E, G>0$

$$Q^4-(A+F+H+D)Q^3-(DH+AD+FD+AH+FH+AF-BE-DI)Q^2$$
$$+(DFI+DHI-ADH-DFH-AFD+BED-AFH+BEH)Q$$
$$-DFHI+ADFH-BDEH=0$$

$$|a_1|=-(A+F+H+D)>0$$

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}=a_1a_2-a_0a_3$$
$$=-(A+F+D+H)(DH+AD+DF+AH+FH+AF-BE-DI)$$
$$-DFI-DHI+ADH+DFH+ADF-BDE+AFH-BEH$$
$$=A^2D+A^2H+A^2F-ABE-ADI+2ADF+DF^2+2AFH+F^2H+AF$$
$$-BEF-DFI+DH^2+2ADH+2DFH+AH^2+FH^2+AFH-BEH$$
$$-DHI+D^2H+AD^2+D^2F+ADH+DFH+ADF-D^2I>0$$

We put this value as U.

$$\begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}=a_3(a_1a_2-a_0a_3)-a_1^2a_4$$
$$=(DFI+DHI-ADH-DFH-ADF+BDE-AFH+BEH)\times U$$
$$-(A+F+H+I)^2(-DFHI+ADFH-BDEH)$$

By some manipulations, the sign of this determinant is proved to be positive. Let this value be V.

$$\begin{vmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}=a_4V>0$$

$$a_4=-DFHI+ADFH-BDEH$$
$$=-\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})LiPf''(N)\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})$$
$$+\left(-(C_{\frac{M}{P}}-I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})LiPf''\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})-BDEH\right)$$
$$=-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}LiPf''(N)\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})-BDEH>0$$

Therefore we could prove that this system as a whole is stable.

VI

Under full employment level, instead of equation (24), we have $Pf'(N)-W_0=0$, and the differential of this equation is,

$$f'(N)dP+Pf''(N)dN=0 \qquad (28)'$$

Therefore, new system shall be (26), (27), (28)′, (29).

By the same method used in section V, the following can be obtained.

$$\frac{dP}{dM}=\begin{vmatrix} 0 & Ci+Ii & -f'(N)(1-g-C_{f(N)}-I_{f(N)}) & \frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) \\ 1 & Li & L_{Pf(N)}Pf'(N) & 0 \\ 0 & 0 & f''(N)-S'(N) & 0 \\ 0 & 0 & 0 & \frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) \end{vmatrix}$$

$$=\frac{1}{D'}\Big(-(Ci+Ii)(f''(N)-S'(N)\Big)\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})>0$$

where

$$D'=\begin{vmatrix} -(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) & Ci+Ii & -f'(N)(1-g-C_{f(N)}-I_{f(N)}) & \frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) \\ L_{Pf(N)}f(N) & Li & L_{Pf(N)}Pf'(N) & 0 \\ f'(N) & 0 & Pf''(N) & 0 \\ \frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) & 0 & 0 & \frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) \end{vmatrix}$$

$$=-\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\Big(LiPf''(N)\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\Big)+Pf''(N)L_{Pf(N)}f(N)(Ci+Ti)\Big)$$

$$+\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\Big(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})LiPf''+(Ci+Ii)LiPf'$$

$$+f(N)(1-g-C_{f(N)}-I_{f(N)})Lif'\Big)>0$$

$$\frac{di}{dM}=\frac{1}{D_1'}\Big\{-\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})Pf''(N)\frac{R}{P}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})$$

$$+\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\Big(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})Pf''(N)-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{1}{P}LiPf'(N)\Big)$$

$$+\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})(f'(N)(1-g-C_{f(N)}-I_{f(N)})f'(N)+Pf''(N)L_{Pf(N)}f(N)(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{1}{P}\Big\}$$

We can not decide the sign of this equation.

$$\frac{dN}{dM}=\frac{1}{D_1'}\Big(\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})(Ci+Ii)f'(N)+f'(N)\Big(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{1}{P}Lif'(N)\Big)>0$$

$$\frac{dR}{dM}=\frac{1}{D_1'}\Big((C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{1}{P}LPf''(N)\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})+(Ci+Ii)\,Pf''(N)\frac{R}{P}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\Big)<0$$

With regard to the balance of payment B, we get,

$$\frac{dP}{dB}=\frac{1}{D_1'}\Big(-\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})Li(f''-S')\Big)>0$$

$$\frac{di}{dB}=\frac{1}{D_1'}\Big(-\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})Pf''(N)L_{Pf(N)}f(N)\Big)>0$$

$$\frac{dN}{dB}=\frac{1}{D_1'}\Big(\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})f'(N)Li\Big)>0$$

$$\frac{dR}{dB}=\frac{1}{D_1'}\Big(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})LiPf''+(Ci+Ii)LiPf'(N)\Big)$$
$$+\frac{1}{D_1'}(f'(N)(1-g-C_{f(N)}-I_{f(N)})Lif'(N)-Pf''(N)L_{Pf(N)}f(N)(Ci+Ii))<0$$

Solving for government expenditures,

$$\frac{dP}{dg}=\frac{1}{D_1'}\Big(-f(N)\,LiPf''(N)\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\Big)>0$$
$$\frac{di}{dg}=\frac{1}{D_1'}\Big(\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})-f(N)L_{Pf(N)}Pf'(N)f'(N)\Big)>0$$
$$\frac{dN}{dg}=\frac{1}{D_1'}\Big(\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})f'(N)L_{Pf(N)}(N)(1-g-C_{f(N)}-I_{f(N)})\Big)>0$$
$$\frac{dR}{dg}=\frac{1}{D_1'}\Big(f(N)\,LiPf''(N)\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\Big)<0$$

And concerning P,

$$\frac{dP}{dp}=\frac{di}{dp}=\frac{dN}{dp}=0$$
$$\frac{dR}{dp}=\frac{1}{D_1'}\Big(-\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})LiPf''\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})-\Big(-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})LiPf''\Big)\Big)$$
$$+\frac{1}{D_1'}\,\frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})(f'(N)(1-g-C_{f(N)}-I_{f(N)})Lif'(N)$$
$$-Pf''(N)\,L_{Pf(N)}f(N)\,(Ci+Ii))>0$$

The characteristic equation by the technique of dynamization of this new open system is given,

$$\begin{vmatrix}-(C_{\frac{M}{P}}+I_{\frac{M}{P}})\frac{M_0}{P^2}+\frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})-Q & Ci+Ii & -f'(N)(1-g-C_{f(N)}-I_{f(N)}) & \frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\\ L_{Pf(N)}f(N) & Li-Q & L_{Pf(N)}Pf(N) & 0\\ f'(N) & 0 & Pf''(N)-Q & 0\\ \frac{R}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}}) & 0 & 0 & \frac{P}{p}(E_{\frac{PR}{p}}-T_{\frac{PR}{p}})\end{vmatrix}=0$$

We change this determinant like follows.

$$\begin{vmatrix}A-Q & B & C & D\\ E & F-Q & G & 0\\ K & 0 & H-Q & 0\\ I & 0 & 0 & D-G\end{vmatrix}=0$$
$$=Q^4-Q^3(A+F+H+D)+Q^2(DH+AD+DF+AH+FH+AF-BE-CK-DI)$$
$$+Q(DFI+DHI-ADH-DFH-ADF+BDE-AFH-BGK+CFK+CDK+BEH)$$
$$-DFHI+ADFH+BDGK-CDFK-BDEH=0$$

$$|\,a_1\,|=-(A+F+H+D)>0$$
$$\begin{vmatrix}a_1 & a_3\\ a_0 & a_2\end{vmatrix}=a_1a_2-a_0a_3$$
$$=-(A+F+H+D)(DH+AD+DF+AH+FH+AF-BE-CK-DI)$$
$$-DFI-DHI+ADH+DFH+ADF-BDE+AFH+BGK-CFK-CDK-BEH$$
$$=-(A^2D+A^2H+A^2F-ABE-ACK-ADI+2ADF+DF^2+2AFH+F^2H+AF^2$$
$$-BEF-CFK-DFI+DH^2+2ADH+2DFH+AH^2+FH^2+AFH-BEH-CHK$$
$$-DHI+D^2H+AD^2+D^2F+ADH+DFH+ADF-BDE-CDK-D^2I-BGK)>0$$

Here we can not decide the sign of this equation. But, if we can assume that $Ci+Ti=0$, the only negative sign that is $-BGK$ would be zero, therefore $a_1a_2-a_0a_3>0$. Let this value be U'.

$$\begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3(a_1a_2 - a_0a_3) - a_1^2a_4$$

$$= U' \times (DFI + DHI - ADH - DFH - ADF + BDE - AFH - BGK + CFK + CDK + BEH)$$
$$- (A^2 + F^2 + H^2 + D^2 + 2AF + 2FH + 2DH + 2AD)$$
$$\times (-DFHI + ADFH + BDGK - CDFK - BDEH)$$

We put this value as V''. After some calculation we can prove that the sign of this equation is positive with the same assumption used just before.

$$\begin{vmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix} = a_4V' > 0$$

$$(\because \ a_4 = -DFHI + ADFH + BDGK - CDFK - BDFH > 0)$$

As a result, we could show the stability of this system of Keynesian model under full employment with rather strict assumption.

VII

As summary, first, we have analyzed the separate effects of change in the exogeneous variables on price, the rate of interest, labor, and the rate of exchange by the techniques of comparative statics based on a very simple aggregate model of the economy. The conclusions are shown in the following table totally.

Secondly, we have analyzed the stability of systems with two, three, four markets at the same time. The stability of each system could be proved.

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	Closed system		Open system	
	Under F. E.	Over F. E.①	Under F. E.	Over F. E.
$\frac{dP}{dM}$	+	+	+	+
$\frac{di}{dM}$	−	−	−	−
$\frac{dN}{dM}$	+	0	+	0
$\frac{dR}{dM}$		②	−	−
$\frac{dP}{dg}$	+	+	+	+
$\frac{di}{dg}$	+	+	+	+
$\frac{dN}{dg}$	+	0	+	0
$\frac{dR}{dg}$			−	−
$\frac{dP}{dB}$			+	+
$\frac{di}{dB}$			+	+
$\frac{dN}{dB}$		③	0	+
$\frac{dR}{dB}$			−	−
$\frac{dP}{dp}$			0	0
$\frac{di}{dp}$			0	0
$\frac{dN}{dp}$			0	0
$\frac{dR}{dp}$			+	+

① F. E. is full employment.
② Flexible exchange rate is assumed.
③ That R is flexible and $B \neq 0$ is assumed

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